Higher Derivatives

1. $y = \left(x + \sqrt{x^2+1}\right)^p$

$$\frac{dy}{dx} = p\left(x+\sqrt{x^2+1}\right)^{p-1}\left(1+\frac{2x}{2\sqrt{x^2+1}}\right) = \frac{p}{\sqrt{x^2+1}}\left(x+\sqrt{x^2+1}\right)^p = \frac{py}{\sqrt{x^2+1}} \qquad \text{....(1)}$$

$$\frac{d^2y}{dx^2} = \frac{p\sqrt{x^2+1}\frac{dy}{dx} - py\frac{x}{\sqrt{x^2+1}}}{x^2+1} = \frac{p\sqrt{x^2+1}\frac{py}{\sqrt{x^2+1}} - x\frac{dy}{dx}}{x^2+1}, \qquad \text{by (1)}$$

$$\therefore (1+x^2)\frac{d^2y}{dx^2} = p^2y - x\frac{dy}{dx} \qquad \Rightarrow (1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - p^2y = 0 \qquad \text{....(2)}$$

Differentiate (2) n times by Leibnitz's theorem,

$$[\,(1+x^2)\,y^{(n+2)} + n\,(2x)\,y^{(n+1)} + (\,n(n-1)/2)\,y^{(n)}\,] + [\,x\,y^{(n+1)} + ny^{(n)}\,] - p^2\,y^{(n)} = 0$$

$$\therefore \quad (1+x^2)\,y^{(n+2)} + (2n+1)\,x\,y^{(n+1)} + (n^2-p^2)\,y^{(n)} = 0$$

2. Let P(n) : $\frac{d^ny}{dx^n} = (-1)^n n! \cos\,(n+1)\theta \sin^{n+1}\theta$, where $x = \cot\theta$.

$$y = \frac{x}{1+x^2} = \frac{\cot\theta}{1+\cot^2\theta} = \frac{\cot\theta}{\csc^2\theta} = \frac{\cos\theta}{\sin\theta}\times\sin^2\theta = \sin\theta\cos\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta}\frac{d\theta}{dx} = \frac{d}{d\theta}\sin\theta\cos\theta \Big/ \frac{d}{d\theta}\cot\theta = \frac{\cos^2\theta - \sin^2\theta}{-\csc^2\theta} = -\cos 2\theta \sin^2\theta = (-1)^1 1! \sin^{1+1}\theta$$

$\therefore$ P(1) is true.

Assume P(k) is true for some $k\in\mathbf{N}$. i.e. $y^{(k)} = (-1)^k k! \cos\,(k+1)\theta \sin^{k+1}\theta$ ....(1)

For P(k+1), $y^{(k+1)} = \frac{d}{dx}y^{(k)} = \frac{d}{d\theta}y^{(k)}\frac{d\theta}{dx} = \frac{d}{d\theta}\left[(-1)^k k! \cos\,(k+1)\theta \sin^{k+1}\theta\right] \Big/ \frac{d}{d\theta}\cot\theta$

$$= (-1)^k k! \left\{\cos\,(k+1)\theta (k+1)\sin^k\theta\cos\theta - (k+1)[-\sin(k+1)\theta]\sin^{k+1}\theta\right\} \Big/ -\csc^2\theta$$

$$= (-1)^{k+1}(k+1)!\left\{\cos\,(k+1)\theta\cos\theta + \sin(k+1)\theta\sin\theta\right\}\sin^{k+2}\theta$$

$$= (-1)^{k+1}(k+1)!\cos\,(k+2)\theta\sin^{k+2}\theta$$

$\therefore$ P(k+1) is true.

By the Principle of Mathematical Induction, P(n) is true $\forall n\in\mathbf{N}$.

3. $y = \left(\sin^{-1}x\right)^2 \quad \Rightarrow \frac{dy}{dx} = 2\left(\sin^{-1}x\right)\frac{1}{\sqrt{1-x^2}} \quad \Rightarrow \sqrt{1-x^2}\frac{dy}{dx} = 2\left(\sin^{-1}x\right)$

$$\Rightarrow \sqrt{1-x^2}\frac{d^2y}{dx^2} + \frac{dy}{dx}\frac{-2x}{\sqrt{1-x^2}} = 2 \;\Rightarrow \frac{1}{\sqrt{1-x^2}}(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 2 \qquad \text{....(1)}$$

Differentiate (1) n times by Leibnitz's theorem,

$$[\,(1-x^2)\,y^{(n+2)} + n\,(-2x)\,y^{(n+1)} + (\,n(n-1)/2)\,(-2)y^{(n)}\,] - [\,x\,y^{(n+1)} + ny^{(n)}\,] = 0$$

$$\therefore \quad (1-x^2)\,y_{n+2} - x\,(2n+1)\,y_{n+1} - n^2\,y_n = 0$$

4. $y = (x^2-1)^n \quad \Rightarrow \quad y_1 = n(x^2-1)^{n-1}(2x) \quad \Rightarrow \quad (x^2-1)\,y_1 = 2n\,(x^2-1)^n = 2nxy$

$\therefore \quad (x^2 - 1)\, y_2 + (2x)\, y_1 = 2nxy_1 + 2ny \quad \Rightarrow \quad (x^2 - 1)\, y_2 + (2x)(1 - n)\, y_1 - 2ny = 0$ ....(1)

Differentiate (1) n times by Leibnitz's theorem,

$[\,(x^2 - 1)\, y_{n+2} + n(2x)\, y_{n+1} + n(n - 1)/2 \quad (2)\, y_n\,] + [2x\,(1 - n)\, y_{n+1} + n\, 2(1 - n)\, y_n\,] - 2ny_n = 0$

$\therefore \quad (x^2 - 1)y_{n+2} + 2xy_{n+1} - n(n + 1)y_n = 0.$ ....(2)

Now, for the given equation: $(1 - x^2)y_2 + 2xy_1 - n(n + 1)y = 0$ ....(3)

Differentiate (3) n times by Leibnitz's theorem,

$[(1 - x^2)y_{n+2} + n(-2x)\, y_{n+1} + n(n - 1)/2 \quad (-2)\, y_n\,]\ n\ (2)\, y_n\,] + n(n + 1)y_n = 0.$

$\therefore \quad (1 - x^2)y_{n+2} - 2x\,(n + 1)\, y_{n+1} = 0$ ....(4)

$$P = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left\{(x^2 - 1)^n\right\} \quad \Rightarrow 2^n n!\, P = y_n$$

$\therefore$ From (4), $(1 - x^2)\dfrac{d^2}{dx^2}\left(2^n n!\, P\right) - 2x(n+1)\dfrac{d}{dx}\left(2^n n!\, P\right) = 0$

$(1 - x^2)\, P'' - 2x\,(n + 1)\, P' = 0$ ....(5)

Hence P satisfies (5) and hence (4) and (3).

5. **(i)** $y = (1 - x^2)^{1/2} \sin^{-1} x$

$$\frac{dy}{dx} = \frac{-2x}{2\sqrt{1 - x^2}}\left(\sin^{-1} x\right) + \sqrt{1 - x^2}\,\frac{1}{\sqrt{1 - x^2}} \qquad \Rightarrow \left(1 - x^2\right)\frac{dy}{dx} = -x\sqrt{1 - x^2}\left(\sin^{-1} x\right) + \left(1 - x^2\right)$$

$$\Rightarrow \left(1 - x^2\right)\frac{dy}{dx} = -xy + \left(1 - x^2\right) \qquad \Rightarrow (1 - x^2)\frac{dy}{dx} + xy = 1 - x^2 \qquad \text{....(1)}$$

(ii) Differentiate (1) w.r.t. x n times by Leibnitz's theorem,

$[\,(1 - x^2)\, y_{n+1} - n\,(-2x)\, y_n + (\, n(n - 1)/2)\,(-2)y_{n-1}\,] - [\, x\, y_n + n(1)\, y_{n-1} \quad] = 0$

$$\therefore \quad (1 - x^2)\frac{d^{n+1}y}{dx^{n+1}} - (2n - 1)x\frac{d^n y}{dx^n} - n(n - 2)\frac{d^{n-1}y}{dx^{n-1}} = 0, \text{ where } n > 2.$$

6. **(i)** $y = \dfrac{x}{x^2 - a^2} = \dfrac{1}{2}\left\{\dfrac{1}{x - a} + \dfrac{1}{x + a}\right\} \quad \Rightarrow y_n = \dfrac{1}{2}(-1)^n (n!)\left\{\dfrac{1}{(x - a)^{n+1}} + \dfrac{1}{(x + a)^{n+1}}\right\}$

(ii) $y^2 = \left(\dfrac{x}{x^2 - a^2}\right)^2 = \dfrac{1}{4}\left\{\dfrac{1}{(x - a)^2} + \dfrac{2}{(x - a)(x + a)} + \dfrac{1}{(x + a)^2}\right\} = \dfrac{1}{4}\left\{\dfrac{1}{(x - a)^2} + \dfrac{1/a}{x - a} - \dfrac{1/a}{x + a} + \dfrac{1}{(x + a)^2}\right\}$

$$= \frac{1}{4a}\left\{\frac{a}{(x - a)^2} + \frac{1}{x - a} - \frac{1}{x + a} + \frac{a}{(x + a)^2}\right\}$$

$$\therefore \quad y_n = \frac{1}{4a}\left\{\frac{(-1)^n (n + 1)!\, a}{(x - a)^{n+2}} + \frac{(-1)^n n!}{(x - a)^{n+1}} - \frac{(-1)^n n!\, a}{(x + a)^{n+1}} + \frac{(-1)^n (n + 1)!\, a}{(x + a)^{n+2}}\right\} = \frac{(-1)^n n!}{4a}\left\{\frac{x + na}{(x - a)^{n+2}} - \frac{x - na}{(x + a)^{n+2}}\right\}$$

7. $y = x^2 \cos x$

$$\frac{y}{x^2} = \cos x \qquad \Rightarrow \frac{1}{x^2}\frac{dy}{dx} - \frac{2y}{x^3} = -\sin x$$

$$\frac{1}{x^2}\frac{d^2y}{dx^2} - \frac{2}{x^3}\frac{dy}{dx} - \left(\frac{2}{x^3}\frac{dy}{dx} - \frac{6y}{x^4}\right) = -\cos y = -\frac{y}{x^2} \quad \Rightarrow x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (x^2 + 6)y = 0 \qquad \text{....(1)}$$

Differentiate (1) w.r.t. x n times by Leibnitz's theorem,

$$\left[x^2y_{n+2}+n(2x)y_{n+1}+\frac{n(n-1)}{2}(2)y_n\right]-\left[4xy_{n+1}+n(4)y_n\right]+\left[\left(x^2+6\right)+n(2x)y_{n-1}+\frac{n(n-1)}{2}(2)y_{n-2}\right]=0$$

$$x^2y_{n+2}+2x(n-2)y_{n+1}+\left(n^2-5n+6+x^2\right)y_n+2nxy_{n-1}+n(n-1)y_{n-2}=0$$

When $x = 0$, $\left(n^2-5n+6\right)y_n+n(n-1)y_{n-2}=0 \Rightarrow (n-2)(n-3)\frac{d^ny}{dx^n}+n(n-1)\frac{d^{n-2}y}{dx^{n-2}}=0$

8. $y = (x-a)^n(x-b)^n \quad \Rightarrow \quad \ln y = n\ln(x-a) + n\ln(x-b)$

$$\frac{1}{y}\frac{dy}{dx}=\frac{n}{x-a}+\frac{n}{x-b}=\frac{n(2x-a-b)}{(x-a)(x-b)} \qquad \Rightarrow (x-a)(x-b)\frac{dy}{dx}=n(2x-a-b)y$$

$$(x-a)(x-b)\frac{d^2y}{dx^2}+(2x-a-b)\frac{dy}{dx}=n(2x-a-b)\frac{dy}{dx}+2ny$$

$$\therefore \quad (x-a)(x-b)\frac{d^2y}{dx^2}-(n-1)(2x-a-b)\frac{dy}{dx}-2ny=0 \qquad \text{....(1)}$$

Differentiate (1) w.r.t. x n times by Leibnitz's theorem, writing $u=\frac{d^ny}{dx^n}$,

$$\left\{(x-a)(x-b)\frac{d^2u}{dx^2}+n(2x-a-b)\frac{du}{dx}+\frac{n(n-1)}{2}(2)u\right\}-(n-1)\left\{2x-a-b)\frac{du}{dx}+n(2)u\right\}-2nu=0$$

$$\therefore \quad (x-a)(x-b)\frac{d^2u}{dx^2}+(2x-a-b)\frac{du}{dx}-n(n+1)u=0$$

9. $y^2 = \sec 2x \quad \Rightarrow \quad 2y\frac{dy}{dx}=2\sec 2x\tan 2x=2y^2\tan 2x \quad \Rightarrow \frac{dy}{dx}=y\tan 2x \quad$ ....(1)

$$\frac{d^2y}{dx^2}=\frac{dy}{dx}\tan 2x+2y\sec^2 2x=(y\tan 2x)\tan 2x+2y\sec^2 2x=y(\tan^2 2x+2\sec^2 2x)=y(3\sec^2 2x-1)$$

$= y(3y^4 - 1)$, by (1) $\qquad \therefore \quad \frac{d^2y}{dx^2}+y=3y^5$

10. $x = c(2\cos\theta + \cos 2\theta)$, $\frac{dx}{d\theta}=c(-2\sin\theta-2\sin 2\theta)=-2c(\sin 2\theta+\sin\theta)=-4c\sin\frac{3\theta}{2}\cos\frac{\theta}{2}$

$y = c(2\sin\theta - \sin 2\theta)$, $\frac{dy}{d\theta}=c(2\cos\theta-2\cos 2\theta)=2c(\cos\theta-\cos 2\theta)=4c\sin\frac{3\theta}{2}\sin\frac{\theta}{2}$

$$\therefore \quad \frac{dy}{dx}=\frac{dy}{d\theta}\Big/\frac{dx}{d\theta}=-\tan\frac{\theta}{2}$$

$$\therefore \quad \frac{d^2y}{dx^2}=\frac{d}{dx}\frac{dy}{dx}=\frac{d}{d\theta}\left(-\tan\frac{\theta}{2}\right)\frac{d\theta}{dx}=\frac{-\frac{1}{2}\sec^2\frac{\theta}{2}}{-4c\sin\frac{3\theta}{2}\cos\frac{\theta}{2}} \qquad \Rightarrow 8c\frac{d^2y}{dx^2}=\csc\frac{3}{2}\theta\sec^3\frac{1}{2}\theta$$

11. **(a)** $y = \lambda^m + \lambda^{-m}$, $x = \lambda + \lambda^{-1} \quad \Rightarrow \quad \frac{dy}{d\lambda}=\frac{m}{\lambda}\left(\lambda^m-\lambda^{-m}\right), \qquad \frac{dx}{d\lambda}=\frac{1}{\lambda}\left(\lambda-\lambda^{-1}\right)$

$$\therefore \quad \left(\frac{dy}{d\lambda}\right)^2=\frac{m^2}{\lambda^2}\left(\lambda^{2m}-2+\lambda^{-2m}\right)=\frac{m^2}{\lambda^2}\left(\lambda^{2m}+2+\lambda^{-2m}-4\right)=\frac{m^2}{\lambda^2}\left(y^2-4\right)$$

$$\left(\frac{dx}{d\lambda}\right)^2 = \frac{1}{\lambda^2}\left(\lambda^2 - 2 + \lambda^{-2}\right) = \frac{1}{\lambda^2}\left(\lambda^2 + 2 + \lambda^{-2} - 4\right) = \frac{1}{\lambda^2}\left(x^2 - 4\right)$$

$$\therefore \quad \left(x^2 - 4\right)\left(\frac{dy}{dx}\right)^2 = \left(x^2 - 4\right)\left(\frac{dy}{d\lambda}\right)^2 \Big/ \left(\frac{dx}{d\lambda}\right)^2 = m^2\left(y^2 - 4\right) \qquad \text{….(1)}$$

(b) Differentiate (1) w.r.t x, $\left(x^2 - 4\right)2\left(\frac{dy}{dx}\right)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)(2x) = m^2(2y)\left(\frac{dy}{dx}\right)$

$$\therefore \quad (x^2 - 4)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - m^2y^2 = 0$$

12. $x = \cos\theta$, $y = \cos^2 p\theta \quad \Rightarrow \quad \frac{dx}{d\theta} = -\sin\theta, \quad \frac{dy}{d\theta} = -2p\cos p\theta \sin p\theta$

$$\therefore \quad \left(\frac{dx}{d\theta}\right)^2 = \sin^2\theta = 1 - x^2, \qquad \left(\frac{dy}{d\theta}\right)^2 = 4p^2\cos^2 p\theta \sin^2 p\theta = 4p^2y\left(1 - y^2\right)$$

$$\left(\frac{dy}{dx}\right)^2 = \left(\frac{dy}{d\theta}\right)^2 \Big/ \left(\frac{dx}{d\theta}\right)^2 = \frac{4p^2y - 4py^2}{1 - x^2} \quad \Rightarrow \left(1 - x^2\right)\left(\frac{dy}{dx}\right)^2 = 4p^2y - 4py^2$$

$$\therefore \quad \left(1 - x^2\right)2\left(\frac{dy}{dx}\right)\frac{d^2y}{dx^2} - 2x\left(\frac{dy}{dx}\right)^2 = 4p^2\left(\frac{dy}{dx}\right) - 8p^2y\left(\frac{dy}{dx}\right) \quad \Rightarrow (1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 4p^2y - 2p^2 = 0 \text{ ….(1)}$$

Differentiate (1) w.r.t. x n times by Leibnitz's theorem,

$[\, (1 - x^2)\, y_{n+2} - n\, (-2x)\, y_{n+1} + (\, n(n-1)/2)\, (-2)y_n \,] - [\, x\, y_{n+1} + n(1)\, y_{n-1} \,] + 4p^2\, y_n = 0$

$$\therefore \quad (1 - x^2)\frac{d^{n+2}y}{dx^{n+2}} - (2n+1)x\frac{d^{n+1}y}{dx^{n+1}} + (4p^2 - n^2)\frac{d^ny}{dx^n} = 0$$

13. Let $P(n) : \frac{d^n}{dx^n}\left(\frac{e^x}{x}\right) = (-1)^n n!\frac{e^x}{x^{n+1}}F_n(-x)$, where $F_n(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + ... + \frac{x^n}{n!}$.

For P(1), $\frac{d}{dx}\left(\frac{e^x}{x}\right) = \frac{xe^x - e^x}{x^2} = (-1)\frac{e^x}{x^2}\left(1 - \frac{x}{1!}\right) = (-1)^1\frac{e^x}{x^{1+1}}F_1(-x)$ $\therefore$ P(1) is true.

Assume P(k) is true for some $k \in \mathbf{N}$, i.e. $\frac{d^k}{dx^k}\left(\frac{e^x}{x}\right) = (-1)^k k!\frac{e^x}{x^{k+1}}F_k(-x)$ ….(1)

For P(k+1), $\frac{d^{k+1}}{dx^{k+1}}\left(\frac{e^x}{x}\right) = (-1)^k k!\, e^x\left[\frac{F_k(-x)}{x^{k+1}} - \frac{k+1}{x^{k+2}}F_k(-x) + \frac{1}{x^{k+1}}F_k'(-x)\right]$

$$= (-1)^{k+1}(k+1)!\frac{e^x}{x^{k+2}}\left[F_k(-x) - \frac{F_k'(-x) + F_k(-x)}{k+1}x\right] \qquad \text{….(2)}$$

$$F_k(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + ... + \frac{x^k}{n!} \Rightarrow F_k'(-x) = -F_{k-1}(-x) \Rightarrow F_k'(-x) + F_k(-x) = F_k(-x) - F_{k-1}(-x) = (-1)^k\frac{x^k}{k!} \text{ ….(3)}$$

Subst. (3) in (2), $\frac{d^{k+1}}{dx^{k+1}}\left(\frac{e^x}{x}\right) = (-1)^{k+1}(k+1)!\frac{e^x}{x^{k+2}}F_{k+1}(-x)$ $\therefore$ P(k+1) is also true.

By the Principle of Mathematical Induction, P(n) is true $\forall n \in \mathbf{N}$.

Replace x by ix and n by 2n in the previous part, $\frac{d^{2n}}{dx^{2n}}\left(\frac{e^{ix}}{ix}\right) = (-1)^n(2n)!\frac{e^{ix}}{(ix)^{2n+1}}F_{2n}(-x)$

$\therefore \quad \frac{d^{2n}}{dx^{2n}}\left(\frac{e^{ix}}{x}\right) = i(-1)^n \frac{(2n)!}{(ix)^{2n+1}}(\cos x + i\sin x)\left(1 - \frac{ix}{1!} + \frac{(ix)^2}{2!} - ... + \frac{(ix)^{2n}}{(2n)!}\right)$

$= \frac{(2n)!}{x^{2n+1}}(-1)^n(\cos x + i\sin x)\left\{\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ... + (-1)^n\frac{x^{2n}}{(2n)!}\right) - i\left(\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - ... + (-1)^{2n-1}\frac{x^{2n-1}}{(2n-1)!}\right)\right\}$

$= \frac{(2n)!}{x^{2n+1}}(-1)^n(\cos x + i\sin x)\{C_{2n}(x) - iS_{2n-1}(x)\}$

Compare imaginary part on both sides, result follows.

14. $T_m(x) = \frac{1}{2^{n-1}}\cos(m\cos^{-1}x) \Rightarrow T_m'(x) = \frac{1}{2^{n-1}}\left[-\sin(m\cos^{-1}x)\right]\frac{-m}{\sqrt{1-x^2}} = \frac{m}{2^{n-1}\sqrt{1-x^2}}\sin(m\cos^{-1}x)$

$\therefore \quad \sqrt{1-x^2}\,T_m'(x) = \frac{m}{2^{n-1}}\sin(m\cos^{-1}x)$

$\sqrt{1-x^2}\,T_m''(x) + \frac{-2x}{\sqrt{1-x^2}}T_m'(x) = \frac{m}{2^{n-1}}\left[\cos(m\cos^{-1}x)\right]\frac{-m}{\sqrt{1-x^2}} = \frac{m^2}{\sqrt{1-x^2}}T_m(x)$

$\therefore \quad (1-x^2)\,T_m''(x) - x\,T_m'(x) + m^2\,T_m(x) = 0$

15. Let $u = (x^2-1)^m$, $u' = m(x^2-1)^{m-1}(2x)$ $\therefore$ $(x^2-1)u' = 2mxu$ ….(1)

Differentiate (1) w.r.t. x (m+1)- times by Leibnitz's theorem,

$(x^2-1)u^{(m+2)} + (m+1)(2x)\,u^{(m+1)} + [(m+1)m/2]\,(2)\,u^{(m)} = 2mx\,u^{(m+1)}$

$\therefore \quad (x^2-1)u^{(m+2)} - 2x\,u^{(m+1)} + m(m+1)\,u^{(m)} = 0$

$$(1-x^2)\left[\frac{1}{2^m m!}u\right]^{(m+2)} - 2x\left[\frac{1}{2^m m!}u\right]^{(m+1)} + m(m+1)\left[\frac{1}{2^m m!}u\right]^{(m)} = 0$$

$\therefore \quad (1-x^2)\,P_m''(x) - 2x\,P_m'(x) + m(m+1)\,P_m(x) = 0.$

16. (i) $y = \sin\ln(1+x) \quad \Rightarrow \quad y_1 = [\cos\ln(1+x)]\frac{1}{1+x} \Rightarrow (1+x)y_1 = \cos\ln(1+x)$

$\therefore \quad (1+x)y_2 + y_1 = -\sin\ln(1+x)\frac{1}{1+x} = -\frac{1}{1+x}y \quad \Rightarrow (1+x)^2y_2 + (1+x)y_1 + y = 0$ ….(1)

(ii) Differentiate (1) w.r.t. x n- times by Leibnitz's theorem,

$$\left[(1+x)^2y_{n+2} + n\cdot 2(1+x)y_{n+1} + \frac{n(n-1)}{2}2y_n\right] + [(1+x)y_{n+1} + ny_n] + y_n = 0$$

$\therefore \quad (1+x)^2y_{n+2} + (2n+1)(1+x)y_{n+1} + (n^2+1)y_n = 0$

17. $y_0 = e^{x^2} \Rightarrow y_1 = 2xe^{x^2} = 2xy$ ….(1)

Differentiate (1) w.r.t. x n- times by Leibnitz's theorem,

$y_{n+1} = 2xy_n + n(2)\,y_{n-1} \quad \Rightarrow \quad y_{n+1} - 2xy_n - 2n\,y_{n-1} = 0$ ….(2)

Let P(r) : $\frac{d^r u_n}{dx^r} = 2^r n(n-1)...(n-r+1)u_{n-r}$, where $u_n = e^{-x^2}y_n$ for $0 \le r \le n$

For P(1), $\frac{du_n}{dx} = \frac{d}{dx}\left(e^{-x^2}y_n\right) = e^{-x^2}y_{n+1} - 2xe^{-x^2}y_n = 2ne^{-x^2}y_{n-1} = 2^1 nu_{n-1}$, by (2).

$\therefore$ P(1) is true.

Assume P(k) is true for some $k\in\mathbf{N}$. i.e. $\frac{d^k u_n}{dx^k} = 2^k n(n-1)...(n-k+1)u_{n-k}$ ....(3)

For P(k + 1),

$$\frac{d^{k+1}u_n}{dx^{k+1}} = \frac{d}{dx}\left(\frac{d^k u_n}{dx^k}\right) = \frac{d}{dx}2^k n(n-1)...(n-k+1)u_{n-k} \quad \text{, by (3).}$$

$$= 2^k n(n-1)...(n-k+1)\frac{d}{dx}e^{-x^2}y_{n-k} = 2^k n(n-1)...(n-k+1)\left[e^{-x^2}y_{n-k+1} - 2xe^{-x^2}y_{n-k}\right]$$

$$= 2^k n(n-1)...(n-k+1)\left[e^{-x^2}(y_{n-k} + 2(n-k)y_{n-k-1}) - 2xe^{-x^2}y_{n-k}\right], \quad \text{by (2)}$$

$$= 2^k n(n-1)...(n-k+1)\left[2(n-k)e^{-x^2}y_{n-k-1}\right] = 2^{k+1}n(n-1)...(n-k+1)(n-k)u_{n-(k+1)}$$

$\therefore$ P(k+1) is true.

By the Principle of Mathematical Induction, P(r) is true $\forall r\in\mathbf{N}$.

When n = r , $\frac{d^n u_n}{dx^n} = 2^n n(n-1)...(n-n+1)u_0 = 2^n n! e^{-x^2} y_0 = 2^n n! e^{-x^2} e^{x^2} = 2^n n!$

18. **(a)** Let P(n) : $\frac{d^n y}{dx^n} = (a^2+b^2)^{\frac{n}{2}} e^{ax}\sin(bx+n\phi)$, where $\cos\phi = \frac{a}{\sqrt{a^2+b^2}}$, $\sin\phi = \frac{b}{\sqrt{a^2+b^2}}$.

For P(1), $\frac{dy}{dx} = \frac{d}{dx}(e^{ax}\sin bx) = e^{ax}(a\sin bx + b\cos bx)$

$$= (a^2+b^2)^{\frac{1}{2}} e^{ax}\left(\sin bx\frac{a}{\sqrt{a^2+b^2}} + \cos bx\frac{b}{\sqrt{a^2+b^2}}\right) = (a^2+b^2)^{\frac{1}{2}} e^{ax}(\sin bx\cos\phi + \cos bx\sin\phi)$$

$= (a^2+b^2)^{\frac{1}{2}} e^{ax}\sin(bx+\phi)$ $\therefore$ P(1) is true.

Assume P(k) is true for some $k\in\mathbf{N}$. i.e. $\frac{d^k y}{dx^k} = (a^2+b^2)^{\frac{k}{2}} e^{ax}\sin(bx+k\phi)$ ....(1)

For P(k+1), $\frac{d^{k+1}y}{dx^{k+1}} = \frac{d}{dx}\left(\frac{d^k y}{dx^k}\right) = \frac{d}{dx}(a^2+b^2)^{\frac{k}{2}} e^{ax}\sin(bx+k\phi)$, by (1)

$$= (a^2+b^2)^{\frac{k}{2}} e^{ax}[a\sin(bx+k\phi) + b\cos(bx+k\phi)]$$

$$= (a^2+b^2)^{\frac{k}{2}+\frac{1}{2}} e^{ax}\left[\frac{a}{\sqrt{a^2+b^2}}\sin(bx+k\phi) + \frac{b}{\sqrt{a^2+b^2}}\cos(bx+k\phi)\right]$$

$$= (a^2+b^2)^{\frac{k+1}{2}} e^{ax}[\cos\phi\sin(bx+k\phi) + \sin\phi\cos(bx+k\phi)] = (a^2+b^2)^{\frac{k+1}{2}} e^{ax}\sin[bx+(k+1)\phi]$$

$\therefore$ P(k+1) is true. By the Principle of Mathematical Induction, P(n) is true $\forall n\in\mathbf{N}$.

(b) Bookwork, omit here.

(c) $y = e^{ax}\sin bx$. Using (b) and (a),

$$\frac{d^n y}{dx^n} = \sum_{k=0}^{n}\binom{n}{k}(e^{ax})^{(n-k)}(\sin bx)^{(k)} = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}e^{ax}b^k\sin\left(bx+\frac{k\pi}{2}\right) = e^{ax}\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k\sin\left(bx+\frac{k\pi}{2}\right)$$

Comparing this result with part (a) and cancelling e^{ax}, we have:

$$(a^2+b^2)^{\frac{n}{2}} e^{ax}\sin(bx+n\phi) = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k\sin\left(bx+\frac{k\pi}{2}\right)$$